

Completing the square and the quadratic formula, **solve any quadratic**

Complete the square and use the quadratic formula.
Worked examples, exam-style questions with answers.

1 Halve, square, take away

2 Square root both sides, keep the $\pm$

3 One formula for every quadratic

4 Factorise, complete the square, or formula

5 Practice + answers

20
EXAM-STYLE
QUESTIONS

What do you do when a quadratic won't factorise?

Completing the square and the quadratic formula feel hard because quadratics don't always factorise. Imagine a square with a corner missing. You fix it by adding and subtracting the same amount.

The picture to keep in your head: The square with a corner missing

Picture $x^2 + 6x$ as tiles: one x -by- x square and six x -by-1 strips. Split the strips into two groups of three and put one group along each side of the square. You almost have a bigger square, $(x + 3)$ by $(x + 3)$, but its corner is missing: a 3-by-3 gap of 9 tiles. So $x^2 + 6x = (x + 3)^2 - 9$: complete the square, then take away the corner you added. The quadratic formula is this same trick done once for every quadratic.

The method, every time

1. Halve the x -term and square it. Add and subtract that amount to keep the equation balanced.
2. Rewrite the equation as a square minus the added amount. This is completing the square.
3. Take the square root of both sides. Remember to keep the $\pm$ symbol for two answers.
4. Use the quadratic formula when factorising fails. It works for any quadratic.
5. Check your answers by substituting back into the original equation.

Words you'll see: **complete the square** = Rewrite a quadratic by adding and subtracting the same amount to form a perfect square. · **quadratic formula** = A formula that solves any quadratic equation using a , b , and c . · **factorise** = Break a quadratic into two brackets that multiply to give the original expression. · **square root** = Find a number that multiplies by itself to give the original number. · **discriminant** = $b^2 - 4ac$: positive means two solutions, zero one, negative none · **discriminant** = $b^2 - 4ac$: positive means two solutions, zero one, negative none

Halve, square, take away

Imagine $x^2 + 6x$ as a square with a corner missing. To complete the square, you halve the x -term ($6 \div 2 = 3$) and make a perfect square $(x + 3)^2$. But this adds 9 tiles, so you must take 9 away. For $x^2 + bx + c$, halve b , square it, then subtract it to keep the equation balanced.

WORKED EXAMPLE Write $x^2 + 6x + 5$ in the form $(x + p)^2 + q$

$$x^2 + 6x + 5$$

Start with the expression

$$(x + 3)^2 = x^2 + 6x + 9$$

Halve 6: the square adds a corner of 9

$$x^2 + 6x = (x + 3)^2 - 9$$

So take the corner away again

$$x^2 + 6x + 5 = (x + 3)^2 - 9 + 5$$

Put the + 5 back

$$(x + 3)^2 - 4$$

Tidy the numbers

$$(x + 3)^2 - 4$$

answer

Watch out

- **Forgetting to subtract the square of half b** Always subtract $(b/2)^2$ after making the square
- **Halving the wrong number** Only halve the coefficient of x , not the constant term

Square root both sides, keep the $\pm$

Imagine $x^2 + 4x - 1$ as a square with a corner missing. To solve it, complete the square, move the number across, then square root both sides. Remember to keep the $\pm$ sign. The turning point of the graph is $(-p, q)$.

WORKED EXAMPLE Solve $x^2 + 4x - 1 = 0$, giving your answers to 2 decimal places

$$x^2 + 4x - 1 = 0$$

Start with the equation

$$(x + 2)^2 - 4 - 1 = 0$$

Complete the square: halve 4, take away 4

$$(x + 2)^2 = 5$$

Move the number across

$$x + 2 = \pm\sqrt{5}$$

Square root both sides, keep the $\pm$

$$x = -2 \pm \sqrt{5}$$

Subtract 2

$$x = 0.24 \text{ or } x = -4.24$$

$\sqrt{5} = 2.236\dots$; round at the end

$$x = 0.24 \text{ or } x = -4.24$$

answer

Watch out

- **Losing the negative root** Always include $\pm$ when taking the square root.
- **Rounding too early** Round only at the end to keep accuracy.

One formula for every quadratic

The quadratic formula is the same trick as completing the square, but done once for every quadratic. Imagine you have a square with a corner missing — the formula fills in that corner and gives you the answers. The formula is x equals negative b plus or minus the square root of b squared minus four a c , all over two a . This works for any quadratic equation.

WORKED EXAMPLE Solve $2x^2 - 3x - 4 = 0$, giving your answers to 2 decimal places

$$2x^2 - 3x - 4 = 0$$

$$a = 2, b = -3, c = -4$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 2 \times (-4)}}{2 \times 2}$$

$$x = \frac{3 \pm \sqrt{9 + 32}}{4}$$

$$x = \frac{3 \pm \sqrt{41}}{4}$$

$$x = \frac{3 \pm 6.4031}{4}$$

$$x = \frac{3 + 6.4031}{4} \text{ or } x = \frac{3 - 6.4031}{4}$$

$$x = 9.4031 / 4 \text{ or } x = -3.4031 / 4$$

$$x = 2.35 \text{ or } x = -0.85$$

Start with the equation

Identify a , b , and c with their signs

Plug into the quadratic formula

Calculate inside the square root

Simplify the square root

Calculate the square root to 4 decimal places

Split into two solutions

Calculate each solution

answer

Watch out

- **Squaring a negative b without brackets, e.g. $-3^2 = -9$** Use brackets: $(-3)^2 = 9$
- **Dividing only the square root by $2a$** Divide the whole numerator by $2a$

Factorise, complete the square, or formula

Imagine a square with a corner missing. Completing the square is like filling in the missing corner, and the quadratic formula is the same trick done automatically. Factorise when numbers work nicely, use the formula when you need decimals or surds, and complete the square when you need a vertex form. The discriminant, $b^2 - 4ac$, tells you how many solutions there are: positive for two, zero for one, and negative for none.

WORKED EXAMPLE How many solutions does $x^2 + 2x + 5 = 0$ have?

$$x^2 + 2x + 5 = 0$$

Start with the equation.

$$b^2 - 4ac = (2)^2 - 4 \times 1 \times 5$$

Calculate the discriminant.

$$4 - 20 = -16$$

Do the arithmetic.

$$-16 < 0$$

Check if the discriminant is negative.

No real solutions

Negative discriminant means no real roots.

none (no real solutions)

answer

Watch out

- **Trying to factorise a quadratic that won't factorise nicely** Use the quadratic formula or complete the square instead.
- **Giving 'no solution' when $b^2 - 4ac$ is zero** One real solution when discriminant is zero.

Exam-style questions

Show your working: most marks are for the method. F = Foundation / IGCSE Core, H = Higher / IGCSE Extended. Answers at the back.

A • COMPLETE THE SQUARE

1 Write $x^2 + 6x + 5$ in the form $(x + p)^2 + q$. H [2]

2 Write $x^2 + 4x + 3$ in the form $(x + p)^2 + q$. H [2]

3 Write $x^2 + 5x + 6$ in the form $(x + p)^2 + q$. H [2]

4 Write $x^2 - 4x + 3$ in the form $(x + p)^2 + q$. H [2]

B • TURNING POINTS

5 Find the turning point of $y = x^2 - 8x + 3$. H [3]

6 Find the minimum value of $y = x^2 + 6x + 5$. H [3]

7 Find the turning point of $y = x^2 + 2x - 15$. H [3]

C • SOLVE BY COMPLETING THE SQUARE

8 Solve $x^2 + 6x + 5 = 0$ by completing the square. H [3]

9 Solve $x^2 + 4x + 3 = 0$ by completing the square. H [3]

10 Solve $x^2 + 6x + 4 = 0$ by completing the square, giving your answers in surd form. H [3]

11 Solve $x^2 - 2x - 7 = 0$ by completing the square, giving your answers to 2 decimal places. H [3]

D • QUADRATIC FORMULA

12 Solve $2x^2 + 4x - 1 = 0$, giving your answers to 2 decimal places. H [3]

13 Solve $3x^2 - 6x + 2 = 0$ using the quadratic formula. H [3]

14 Solve $x^2 = 3x + 5$, giving your answers to 2 decimal places. H [3]

15 Solve $2x^2 + 5x - 3 = 0$ using the quadratic formula. H [3]

16 Solve $3x^2 + 6x + 2 = 0$ using the quadratic formula. H [3]

E • CHOOSE AND EXPLAIN

17 A rectangle has length $(x + 2)$ and width $(x - 1)$. Its area is 24. Find x . H [4]

- 18** For the equation $2x^2 + 8x + 5 = 0$, find the number of real solutions. H [4]
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- 19** Show that the solutions of $2x^2 + 5x - 3 = 0$ are $x = 1/2$ and $x = -3$. H [4]
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- 20** Show that $x^2 + 4x + 7 = 0$ has no real solutions. H [4]
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Marks are a guide to how much working an exam would expect. Questions written by Bloom Education in the style of GCSE (AQA, Edexcel, OCR) and Cambridge IGCSE papers.

Check your work

1 $(x+3)^2 - 4x^2 + 6x + 5 = (x+3)^2 - 9 + 5 = (x+3)^2 - 4$

2 $(x+2)^2 - 1x^2 + 4x + 3 = (x+2)^2 - 4 + 3 = (x+2)^2 - 1$

3 $(x+2.5)^2 - 0.25x^2 + 5x + 6 = (x+2.5)^2 - 6.25 + 6 = (x+2.5)^2 - 0.25$

4 $(x-2)^2 - 1x^2 - 4x + 3 = (x-2)^2 - 4 + 3 = (x-2)^2 - 1$

5 (4, -13) Complete the square: $x^2 - 8x = (x-4)^2 - 16$. So $y = (x-4)^2 - 13$. Turning point is (4, -13).

6 -4 Complete the square: $x^2 + 6x = (x+3)^2 - 9$. So $y = (x+3)^2 - 4$. Minimum value is -4.

7 (-1, -16) Complete the square: $x^2 + 2x = (x+1)^2 - 1$. So $y = (x+1)^2 - 16$. Turning point is (-1, -16).

8 $x = -1$ or $x = -5$ $x^2 + 6x + 5 = 0$. Complete the square: $x^2 + 6x + 9 = 4$. $(x+3)^2 = 4$. $x+3 = \pm 2$. $x = -3 \pm 2$.

9 $x = -1$ or $x = -3$ $x^2 + 4x + 3 = 0$. Complete the square: $x^2 + 4x + 4 = 1$. $(x+2)^2 = 1$. $x+2 = \pm 1$. $x = -2 \pm 1$.

10 $x = -3 \pm \sqrt{5}$ $(x+3)^2 - 9 + 4 = 0$, so $(x+3)^2 = 5$ and $x+3 = \pm\sqrt{5}$.

11 $x = 3.83$ or $x = -1.83$ $(x-1)^2 - 1 - 7 = 0$, so $(x-1)^2 = 8$ and $x = 1 \pm \sqrt{8}$.

12 $x = 0.22$ or $x = -2.22$ $a = 2$, $b = 4$, $c = -1$: $x = (-4 \pm \sqrt{24})/4$.

13 $x = 1.58$ or $x = 0.42$ $a = 3$, $b = -6$, $c = 2$: $x = (6 \pm \sqrt{12})/6$.

14 $x = 4.19$ or $x = -1.19$ Rearrange first: $x^2 - 3x - 5 = 0$. $a = 1$, $b = -3$, $c = -5$: $x = (3 \pm \sqrt{29})/2$.

15 $x = 0.5$ or $x = -3$ $a = 2$, $b = 5$, $c = -3$: $x = (-5 \pm \sqrt{49})/4 = (-5 \pm 7)/4$.

16 $x = -0.42$ or $x = -1.58$ $a = 3$, $b = 6$, $c = 2$. $x = [-6 \pm \sqrt{(6^2 - 4 \times 3 \times 2)}] / (2 \times 3)$. $x = [-6 \pm \sqrt{(36 - 24)}] / 6$. $x = [-6 \pm \sqrt{12}] / 6$. $x = [-6 \pm 2\sqrt{3}] / 6$. $x = -1 \pm (\sqrt{3})/3$. $x \approx -1 \pm 0.577$. $x \approx -1.58$ or -0.42 .

17 $x = 4.62$ $x^2 + x - 26 = 0$ gives $x = (-1 \pm \sqrt{105})/2$. The width $x - 1$ must be positive, so $x = 4.62$.

18 2 real solutions Use discriminant $b^2 - 4ac = 8^2 - 4 \times 2 \times 5 = 64 - 40 = 24$. Since discriminant is positive, there are 2 real solutions.

19 $x = 1/2$ and $x = -3$ Use quadratic formula: $x = [-5 \pm \sqrt{(25 + 24)}] / 4 = [-5 \pm \sqrt{49}] / 4 = [-5 \pm 7] / 4$. So $x = (2)/4 = 1/2$ or $x = (-12)/4 = -3$.

20 $b^2 - 4ac = -12$, which is negative, so no real solutions $a = 1$, $b = 4$, $c = 7$: $4^2 - 4 \times 1 \times 7 = 16 - 28 = -12 < 0$.

Exam checklist

- Rearrange to $= 0$ before starting.
- Write a , b , c with their signs included.
- Use brackets around negative numbers for clarity.
- Keep $\pm$ until the end to avoid mistakes.
- Round only at the end to keep accuracy.
- Check $b^2 - 4ac$ to see how many solutions there are.
- Always check your answers by substitution.

Got stuck on a question?

- Find the section it belongs to and go through its worked example again, line by line.
- Put your answer back into the question. If it doesn't work, look for a sign or a step you skipped.
- Still stuck? Message us on Instagram (@bloom_education_international). We read every message.

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